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Fluctuations in the Thermodynamic Limit of Focussing Cubic Schrödinger

B. Rider[1]

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We consider the statistical mechanics of a complex field Z whose dynamics are governed by the focussing cubic Schrödinger equation. Here the Hamiltonian

$$H = \int_{\Omega} \left[\frac{1}{2} |\nabla Z|^2 - \frac{1}{4} |Z|^4 \right] dx$$

is unbounded from below, preventing the natural Gibbs measure from being normalizable. This difficulty may be circumvented[(5)] by taking Ω the circle of perimeter L and fixing the mean-square (which is conserved by the dynamics): $\int_0^L |Z|^2 \, dx = LD$ for positive "density" D. The resulting (probability) measure on paths is absolutely continuous to the two-dimensional Wiener measure and is known to be invariant under the flow.[(2, 7)] One way to extend this picture to the whole-line flow is to take the thermodynamic limit ($L \uparrow \infty$). Unfortunately, the unboundedness of H causes vast local concentration of the field as L increases and leads to collapse at $L = \infty$.[(11)] Here we attempt to capture fluctuations away from this collapse by performing a joint continuum and infinite-volume limit for an appropriate lattice ensemble. The result is that, for high density, the scaled paths go over into a White Noise.

KEY WORDS: Invariant ensemble; NLS; thermodynamic limit.

1. INTRODUCTION AND RESULTS

The focussing cubic Schrödinger equation taken on the circle of perimeter L may be written

$$-i \frac{\partial}{\partial t} Z = \frac{\partial^2}{\partial x^2} Z + |Z|^2 Z$$

[1] Department of Mathematics, Duke University, Durham, North Carolina 27708; e-mail: rider@math.duke.edu

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for the complex field $Z(x, t) = Q(x, t) + iP(x, t)$. The study of a statistical mechanics for fields of this type was initiated by Lebowitz, Rose, and Speer (LRS)[5, 6] as forming an idealized description of Langmuir waves in a plasma, propagating laser fields in a nonlinear media and like phenomena.

Recognizing that the pair (Q, P) form conjugate variables in the Hamiltonian formalism of the equation with

$$H = \frac{1}{2}\int_0^L [(Q')^2 + (P')^2]\, dx - \frac{1}{4}\int_0^L [Q^2 + P^2]^2\, dx,$$

LRS introduced the canonical Gibbs ensemble for the dynamics:[2]

$$e^{-H}\, d^\infty P\, d^\infty Q = e^{+(1/4)\int_0^L [Q^2+P^2]^2} \times e^{-(1/2)\int_0^L [(Q')^2+(P')^2]}\, d^\infty Q\, d^\infty P. \tag{1}$$

The meaning of this formal object is as follows. The second factor signifies that (Q, P) is the periodic Wiener process, i.e., it is formed of standard 2-d paths starting at $(Q, P)(0) = c$ and conditioned to return to c at $x = L$, this common value being distributed uniformly over the plane. The first part is just a density. While it has a proper sense as paths are continuous under the Wiener measure, the resulting ensemble has infinite total mass. This is remedied by taking a micro-canonical viewpoint in which the measure is restricted by conditioning on the constant of motion $\int_0^L (Q^2 + L^2) = LD$ for fixed $D > 0$. That is, we consider the probability measure on paths with partition function:[3]

$$\mathfrak{Z}_L = \int_{\mathbb{R}^2} \mathbf{E}_c \left[e^{\frac{1}{4}\int_0^L [Q^2(x)+P^2(x)]^2\, dx}, \int_0^L Q^2(x) + P^2(x)\, dx = LD, (Q, P)(L) = c \right] dc.$$

Here $\mathbf{E}_\bullet$ is the mean of the free Wiener process (Q, P) starting at $\bullet \in \mathbb{R}^2$. The fact that now $\mathfrak{Z}_L < \infty$ was first proved by LRS.

The existence of the flow in the present ensemble along with the invariance of the latter was established independently by Bourgain[2] and McKean.[7] McKean also discussed the question of the thermodynamic $(L \uparrow \infty)$ limit: ref. 8 puts forward a proof that the full limit does not exist. That is, the statement was that depending on how the circle is taken to the line, one sees an infinity of Gibbs states. This was offered up as a possible explanation for the differing numerical results of other authors. Simulations of LRS suggested a phase transition: the ensemble living near solitons/radiation

[2] The setup in LRS is more general. They consider nonlinearities beyond the cubic as well as the temperature dependent ensemble in which $e^{-\beta H}$ replaces e^{-H}.

[3] The peculiar but useful notation $(Q, P)(L) = c$ and the like indicate densities: $E[F(Q) = a, G(P) = b] = (\partial^2/\partial N\, \partial M)\, E[F(Q) \leqslant N, G(P) \leqslant M]|_{N=a,\, M=b}$.

for high/low values of D. The work of Burlakov[3] appeared to run counter to this interpretation.

Unfortunately ref. 8, contains an error. In fact, not only does the thermodynamic limit exist, it is trivial: as $L \uparrow \infty$ the ensemble collapses onto the delta measure on the 0-path.[4] This is made rigorous in ref. 11 where the free energy is computed:

$$\lim_{L \uparrow \infty} \frac{1}{L^3} \log \mathfrak{Z}_L = \sup_{\int_{-\infty}^{\infty} |f(x)|^2 \, dx = D} \left\{ \frac{D}{4} \int_{-\infty}^{\infty} |f(x)|^4 \, dx - \frac{1}{2} \int_{-\infty}^{\infty} |f'(x)|^2 \, dx \right\}. \tag{2}$$

For whatever $D > 0$, the leading paths which contribute to (2) live near a *single* soliton of height L and width $1/L$, and it follows that at $L = \infty$ the total energy is concentrated on a (equally distributed on $\mathbb{R}$) point. We note in passing that the above free energy is positive and continuous in D—one does not see evidence of phase transition at this level.

After the fact, the collapse may be viewed as the only way possible for the rather fierce competition between the quartic interaction and the microcanonical fiat to resolve itself. It also indicates that at finite L, the ensemble is supported on very rough paths. One way to capture this is by trying to understand fluctuations about this trivial limit—or see what is happening away from the soliton. This is the question posed here. Since the path goes down to zero, the problem may be formally stated as determining a rate $\gamma_L \uparrow \infty$ and a limit law for the (scaled) field $(\gamma_L Q, \gamma_L P)$ under the microcanonical measure.

Now such a computation in the infinite dimensional diffusion ensemble is beyond us. Therefore, in hope of shedding some light on the matter, we introduce and study an appropriate lattice model of the cubic Schrödinger system. The idea is simple: Wiener measure is replaced by a Gaussian lattice field. Also taking things one-dimensional, we introduce the Hamiltonian:[5]

$$H_{L,\Delta}[Q] = -\frac{1}{4} \sum_{k=0}^{\lceil L/\Delta \rceil - 1} Q_k^4 \Delta + \frac{1}{2} \sum_{k=0}^{\lceil L/\Delta \rceil - 1} \left(\frac{Q_{k+1} - Q_k}{\Delta} \right)^2 \Delta, \tag{3}$$

for periodic fields Q, $Q_0 = Q_{\lceil L/\Delta \rceil}$. In order to best mimic the diffusion ensemble, the lattice spacing Δ is taken to depend on L as in $\Delta = 1/L$ (the particular choice is explained below). In other words, we are taking a joint continuum and infinite-volume limit.

[4] This possibility was discussed by LRS.

[5] From here on we will drop the usage of $\lceil$ and $\rceil$. It will be clear from the context when we are running through integers.

Our object of study is then the measure

$$d\mathbf{M}_L[Q] = \frac{1}{\mathbf{Z}_L} e^{\{-H_{L,1/L}[Q]\}} \delta\left(\frac{1}{L}\sum_{k=0}^{L^2-1} Q_k^2 = LD\right) dQ_0\, dQ_1 \cdots dQ_{L^2-1}: \tag{4}$$

free Gaussian Q subject to a quartic interaction term and then conditioned to remain on the sphere of radius $\sqrt{LD}$. As in the diffusion case, the ensemble $\mathbf{M}_L$ collapses onto $\delta(Q \equiv 0)$ as $L \uparrow \infty$. Now however we are able to determine a non-trivial limit for the field scaled at rate $\sqrt{L}$, at least for large enough density. Our main result is the following.

Theorem 1. For D sufficiently large, the scaled lattice field $Q \to \sqrt{L}\, Q$ decouples into a "White Noise" as $L \uparrow \infty$. In particular, for any finite collection of positions $x_0 < x_1 < \cdots < x_m$: with ρ_D a positive constant to be defined below,

$$\lim_{L\uparrow\infty} \mathbf{M}_L[\sqrt{L}\, Q_{Lx_0} = a_0, \sqrt{L}\, Q_{Lx_1} = a_1, \ldots, \sqrt{L}\, Q_{Lx_m} = a_m]$$

$$= \prod_{k=0}^{m} \frac{\exp(-\rho_D a_k^2)}{\sqrt{\pi/\rho_D}}$$

in the sense of weak convergence of measures. This follows from computing the following limiting joint density

$$\lim_{L\uparrow\infty} \mathbf{M}_L\left[\sqrt{L}\, Q_0 = a, \sqrt{L}\, Q_{Lx} = b, L\left(\sum_{k=1}^{Lx-1} Q_k^2 \frac{1}{L}\right) = I\right]$$

$$= \frac{\exp(-\rho_D a^2)}{\sqrt{\pi/\rho_D}} \times \frac{\exp(-\rho_D b^2)}{\sqrt{\pi/\rho_D}} \times \delta\left(I - \frac{x}{2\rho_D}\right), \tag{5}$$

which has the interpretation as the density of $[Z(0), Z(x), \int_0^x Z^2(x')\, dx']$ in the continuum model.

We believe that this decoupling, or White Noise, in the limit provides some color to roughness of the typical path in the constructed invariant ensemble. That the result is stated for large D only stems from a missing uniqueness statement in a variational problem connected to the lattice Hamiltonian (3). It is our further belief that the above obtains at all D. With that, we will see that the limiting mean-square ρ_D (see (12)) reflects the structure of H and has an interesting behavior as a function of the density D: $\rho_D \uparrow \infty$ with D and $\rho_0 = 0$. So, believing Theorem 1 to hold at all D one would have that fluctuations away from the collapse are increasingly heavy-tailed as $D \downarrow 0$. One explanation of the numerical experiments[3, 5] may be that while solitons persist, the relatively high fluctuations at small D obscures this from view.

The rest of the paper is devoted to proving Theorem 1. We begin in Section 2 with various technicalities. First, the measure $\mathbf{M}_L$ is discussed in greater detail and an auxiliary measure through which $\mathbf{M}_L$ is better studied is introduced. We also state a Lemma on the optimal configurations for maximizing (3) on the sphere as these clearly govern the behavior of $\lim_{L \uparrow \infty} \mathbf{M}_L$. Section 3 computes the limit law of the scaled marginal $\sqrt{L} \times Q_0$. The computation of (5) is completed in Section 4.

2. PRELIMINARIES

2.1. The Measure $\mathbf{M}_L$

Bringing in $\mathbf{E}_\bullet =$ the expectation of the free Gaussian field with initial point $\bullet \in R$, lattice $\Delta = 1/L$ and mean-square $\Delta^{-1} = L$, the ensemble $\mathbf{M}_L$ is expressed more concretely through its partition function

$$\mathbf{Z}_L = \int_{-\sqrt{LD}}^{\sqrt{LD}} \mathbf{E}_c \left[e^{\frac{1}{4L} \sum_0^{L^2-1} Q_k^4}, \sum_{k=0}^{L^2-1} Q_k^2 \frac{1}{L} = LD, Q_{L^2} = c \right] dc.$$

Note the enforced periodic conditions making $\mathbf{M}_L$ rotation invariant. We also introduce the density

$$p(x, a, b, I) = \mathbf{E}_a \left[e^{\frac{1}{4L} \sum_0^{Lx-1} Q_k^4}, \sum_{k=0}^{Lx-1} Q_k^2 \frac{1}{L} = I, Q_{Lx} = b \right], \tag{6}$$

through which many quantities of interest for $\mathbf{M}_L$ can be expressed: e.g., $\mathbf{Z}_L = \int p(L, c, c, LD)\, dc$ and the marginal density is

$$\mathbf{M}_L[a] = \mathbf{M}_L[Q_0 = a] = \mathbf{Z}_L^{-1} p(L, a, a, LD). \tag{7}$$

The arguments of ref. 11 can be adapted to show that $\mathbf{M}_L[a]\, da \to \delta_0$, from which the collapse is evident by rotation invariance. That is not dwelled on here. The present goal is to understand *scaled* quantities: $L^{-1/2} \mathbf{M}_L[L^{-1/2} a]$ and the like.

Before moving on, a word is in order as to our choice $\Delta = 1/L$. The analysis of the diffusion or $\mathfrak{Z}_L$ ensemble rests on the quartic interaction $\int |Z|^4$ leading to paths that want to concentrate locally competing with the energy term $\int |\nabla Z|^2$ which likes things smooth. Now, ignoring for a second the periodicity and uninteresting constant multipliers, in the present lattice setup the total mass $\mathbf{Z}_L$ is seen to have the form:

$$\mathbf{Z}_L \simeq \int_{\Sigma_0^{L^2-1}\sigma_i^2 = LD} \exp\left[\frac{1}{4}\frac{1}{L}\sum_{i=0}^{L^2-1}\sigma_i^4 - \frac{L}{2}\sum_{i=0}^{L^2-2}(\sigma_{i+1}-\sigma_i)^2\right] d\sigma$$

$$= c_{L,D}\int_{S_1^{L^2-1}} \exp\left[L^3 D\left\{\frac{D}{4}\sum_{i=0}^{L^2-1}\sigma_i^4 + \sum_{i=0}^{L^2-2}\sigma_i\sigma_{i+1}\right\}\right.$$

$$\left. + L^{3/2}\sqrt{D}(\sigma_1+\sigma_{L^2-1})\right] d\sigma. \tag{8}$$

Thus the competition between $\sum \sigma_i^4$ (favoring a soliton) and $\sum \sigma_i\sigma_{i+1}$ (favoring radiation) is present here. Also, it is relatively easy to see that $\lim_{L\uparrow\infty} L^{-3}\log \mathbf{Z}_L = \max\{(1/4)\sum_{-\infty}^{\infty}\sigma_k^4 + \sum_{-\infty}^{\infty}(\sigma_k-\sigma_{k+1})^2 \text{ on } \sum_{-\infty}^{\infty}\sigma^2 = D\}$ in which you have the same rate with free energy analogous to that of $\mathfrak{Z}_L$. Finally, as the continuum ensemble concentrates near a single soliton of width $\simeq 1/L$, one should take $\Delta \downarrow 0$ at least as fast in order to "sample" the path at the correct scale.

2.2. The Auxiliary Measure

Examining the exponent in (8), it is clear that in order to understand $\mathbf{M}_L$ one must investigate the quartic form[6]

$$H_{L^2,D}(\sigma) = \frac{D}{4}\sum_{i=-L^2/2}^{L^2/2}\sigma_i^4 + \sum_{i=-L^2/2}^{L^2/2-1}\sigma_i\sigma_{i+1} \tag{9}$$

maximized over $\sum_{i=-L^2/2}^{L^2/2}\sigma_i^2 = 1$. The Lagrange multiplier for this problem will also be important below. It is

$$\lambda_{L^2,D}(\sigma) = D\sum_{-L^2/2}^{L^2/2}\sigma_i^4 + 2\sum_{-L^2/2}^{L^2/2}\sigma_i\sigma_{i+1} = 2H_{L^2,D}(\sigma) + \frac{D}{2}\sum_{-L^2/2}^{L^2/2}\sigma_i^4. \tag{10}$$

Next, the study of the scaled $\mathbf{M}_L$ ensemble is transfered to that of the auxiliary measure μ_{L^2} on the L^2 dimensional sphere

$$d\mu_{L^2}(\sigma_{-L^2/2}, \sigma_{-L^2/2+1}, \ldots, \sigma_{L^2/2})$$

$$= \frac{1}{z_{L^2}}\exp[DL^3 H_{L^2,D}(\sigma)]\,\delta\left(\sum_{-L^2/2}^{L^2/2}\sigma_i^2 = 1\right) d\sigma_{-L^2/2}\cdots d\sigma_{L^2/2} \tag{11}$$

[6] We will usually suppress the dependence of H_{L^2} and λ_{L^2} on D. Note also the re-indexing.

with its own private partition function z_{L^2}. Much of the proof involves re-expressing $\mathbf{M}_L$ averages in terms of $\mathbf{E}^{\mu_{L^2}}$ averages.

The parameter ρ_D appearing as the limiting mean-square in Theorem 1 may now be defined:

$$\rho_D = \sqrt{\lim_{L\uparrow\infty} \mathbf{E}^{\mu_L^2}[\lambda^2_{L^2,D}(\sigma)/4] - 1}, \tag{12}$$

provided the limit exists (which is proved for $D \gg 1$).

2.3. The Lattice Hamiltonian

As mentioned, the variational problem inherent in (9)—maximizing H_{L^2} on the sphere—plays a central role in the sequel. The next lemma states all that we know; the proof is found in the Appendix A.

Lemma 1. Denoting

$$m_{L^2,D} = \max_{\sum_{-L^2/2}^{L^2/2} \sigma_i^2 = 1} H_{L^2,D}(\sigma) \qquad \text{and} \qquad m_{\infty,D} = \sup_{\sigma\in\ell^2} H_{\infty,D}(\sigma),$$

we have concentration of μ_{L^2} as in

$$\mu_{L^2}(H_{L^2,D}(\sigma) < m_{\infty,D} - \varepsilon) \leqslant c_1 \varepsilon^{-L^2} e^{-c_2 L^3 \varepsilon} \tag{13}$$

as well as the following.

(a) For $L\uparrow\infty$ and any $D>0$, $m_{L^2,D}>1$. More precisely we know $m_{L^2,D} \geqslant 1 + D^2/32$ for $D \ll 1$ while $m_{L^2,D} \geqslant D/4 + 7/4D - \mathrm{O}(D^{-3})$ for $D \gg 1$.

(b) For all $D>0$ the maximizing σ resembles a soliton: it is largest at $k=0$ and is increasing/decreasing to the left/right. In fact, the maximizer decays exponentially far out, as in $|\sigma_k| \leqslant c_1 e^{-c_2|k|}$ for $|k| \geqslant M$ for some large M with c_1 and c_2 depending only on D. For large D the decay is sharper as in $\sigma_k \sim D^{-|k|}$. Furthermore, the maximum $m_{\infty,D}$ is attained for all $D>0$.

(c) Modulo the obvious reflection, the maximizer of H is unique for all sufficiently large values of D. A simple but important consequence being that $\lambda_{L^2,D}$ converges to a constant in μ_{L^2}-probability.

Remark. The real barrier to having Theorem 1 for all densities is the fact that we have the above uniqueness statement (part (c)) for large D only. Indeed, the matter of uniqueness is much harder than one would first guess.

2.4. Outline of the Computation

We will start with one-dimensional marginal

$$\mathbf{M}_L[a] = Z_L^{-1} p(L, a, a, LD) = \mathbf{M}_L[0] \frac{p(L, a, a, LD)}{p(L, 0, 0, LD)}.$$

Now the density p has an explicit expression as a spherical integral: to wit,

$$\begin{aligned} &p(L, a, a, LD) \\ &= (L/2\pi)^{L^2/2} (DL^2 - a^2)^{\frac{L^2-2}{2}} \exp(-L^3 D) \exp(a^4/4L) \\ &\times \int_{\sum_{k=1}^{L^2-1} \sigma_k^2 = 1} \exp\left[\frac{1}{4L} (DL^2 - a^2)^2 \sum_{k=1}^{L^2-1} \sigma_k^4 + L(DL^2 - a^2) \sum_{k=1}^{L^2-2} \sigma_k \sigma_{k+1} \right] d\sigma. \end{aligned}$$

This allows us to re-write $\mathbf{M}_L[a]$ in terms of the measure μ_{L^2} defined in (11):

$$\begin{aligned} \mathbf{M}_L[Q_0 \in da] = \mathbf{M}_L[0] &\left(1 - \frac{a^2}{DL^2}\right)^{\frac{L^2-3}{2}} \exp(a^4/4L) \\ &\times \mathbf{E}^{\mu_{L^2}} \left[\exp\left\{ -L\lambda_{L^2}(\sigma) \frac{a^2}{2} \right.\right. \\ &\left.\left. + La\sqrt{L^2 D - a^2} (\sigma_{-L^2/2} + \sigma_{L^2/2}) + \frac{a^4}{4L} \sum \sigma^4 \right\} \right] da. \quad (14) \end{aligned}$$

Next, as tightness obtains ($\mathbf{M}_L[Q_0^2] = D$) the scaled marginal (take a into $\sqrt{L}\, a$) may be shown to satisfy: for a bounded and L large,

$$\mathbf{M}_L[\sqrt{L}\, Q_0 \in da] = \frac{\mathbf{M}_L[0]}{\sqrt{L}} \mathbf{E}^{\mu_{L^2}}[e^{\{-\lambda_{L^2}(\sigma)\, a^2/2 + L^{3/2}\sqrt{D}(\sigma_{-L^2/2} + \sigma_{L^2/2})\, a\}}]\, da \quad (15)$$

up multiplicative errors $1 + O(1/L)$ on the right hand side. Now, μ_{L^2} concentrates sharply at the maximizers of H_{L^2} and Lemma 1(b) shows that the tail variables $(\sigma_{-L^2/2}, \sigma_{L^2/2})$ are exponentially small in L at $H = \max$. Again by the concentration of μ_{L^2} it is natural to hope that λ_{L^2} should be roughly constant for $L \uparrow \infty$, and thus that (15) should settle down to a centered Gaussian with mean-square one over λ_∞; the tail variables being unimportant.

However, this reasoning is just *wrong*. The tail variables *do* figure in: $\sigma_{L^2/2}$ exhibits enough fluctuation (away from zero) that $\mathbf{E}^{\mu_{L^2}}[\exp(L^{3/2}\sqrt{D}\, a\sigma_{L^2/2})]$

is strictly-positive for $L \uparrow \infty$.[7] We will need the following, the proof of which is left to Appendix A.

Lemma 2. There exists some positive γ depending on D such that

$$\limsup_{L \uparrow \infty} \mathbf{E}^{\mu_L^2}[\exp(L^3 \gamma \sigma_{L^2/2}^2)] < \infty.$$

For D sufficiently large this will also imply that $L^{-1/2}\mathbf{M}_L[0] = O(1)$ for $L \uparrow \infty$.

Given Lemma 2 we will show the pair $(L^{3/2}\sigma_{-L^2/2}, L^{3/2}\sigma_{L^2/2})$ has a Gaussian limit for $L \uparrow \infty$. This is the main argument needed to establish convergence of the scaled $\mathbf{M}_L$ marginal and occupies the next section. With the marginal in hand, we then turn (Section 4) to the computation of the limiting scaled joint density (5) central to Theorem 1. That too has an expression in term of the p's:

$$\mathbf{M}_L\left[\sqrt{L}\, Q_0 = a, \sqrt{L}\, Q_{Lx} = b, \sum_{k=0}^{Lx-1} (\sqrt{L}\, Q_k)^2 \frac{1}{L} = I\right]$$
$$= \frac{1}{L^2 \mathbf{Z}_L} p\left(x, \frac{a}{\sqrt{L}}, \frac{b}{\sqrt{L}}, \frac{I}{L} + \frac{a^2}{L^2}\right) p\left(L - x, \frac{b}{\sqrt{L}}, \frac{a}{\sqrt{L}}, DL - \frac{I}{L} - \frac{a^2}{L^2}\right). \tag{16}$$

3. THE SCALED MARGINAL DISTRIBUTION

Our objective here is to establish:

Proposition 1. Let D be large. Then for $L \uparrow \infty$ the law of the tail variables $(L^{3/2}\sqrt{D}\,\sigma_{-L^2/2}, L^{3/2}\sqrt{D}\,\sigma_{L^2/2})$ converges to a pair of independent centered Gaussians with variance $\lambda/2 - \sqrt{\lambda^2/4 - 1}$. Here $\lambda = \lambda_{D,\infty} > 2$. As an immediate corollary one gets the convergence in law of the scaled $\mathbf{M}_L$ marginal as in

$$\lim_{L \uparrow \infty} \mathbf{M}_L[\sqrt{L}\, Q_0 \in da] = \lim_{L \uparrow \infty} \frac{\mathbf{M}_L[0]}{\sqrt{L}} \mathbf{E}^{\mu_L^2}[e^{-\frac{1}{2}\lambda_{L^2}(\sigma) a^2 + \sqrt{D}\, L^{3/2}(\sigma_{-L^2/2} + \sigma_{L^2/2})}]$$
$$= \sqrt{\frac{\rho_D}{\pi}} \exp[-\rho_D a^2]\, da \tag{17}$$

[7] The symmetry of H_{L^2} implies $\sigma_{-L^2/2}$ and $\sigma_{L^2/2}$ are equally distributed.

where $\rho_D = \sqrt{\lambda^2/4-1}$. Note this also pins down the convergence of $L^{-1/2}\mathbf{M}_L[0]$ for $D \gg 1$.

The proof of Proposition 1 requires the following two lemmas. The strategy is to determine an integral equation satisfied by any limiting density function of the tail variables and then search for solutions.

Lemma 3. Under μ_{L^2}, the law of $L^{3/2}\sqrt{D}\,\sigma_{L^2/2}$ is tight for $L \uparrow \infty$ (assuming large D); any limiting density function f is even and solves

$$f(x) = f(0)\exp(-\lambda x^2/2)\int_{-\infty}^{\infty}\exp(xy)\,f(y)\,dy. \tag{18}$$

Likewise, any limiting joint density

$$f(x,y) = \lim_{L'\uparrow\infty} \mu_{L^2}(L^{3/2}\sqrt{D}\,\sigma_{-L^2/2} = x,\ L^{3/2}\sqrt{D}\,\sigma_{L^2/2} = y)$$

satisfies

$$f(x,y) = f(0,0)\exp(-\lambda(x^2+y^2)/2)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp(xz+yw)\,f(z,w)\,dz\,dw \tag{19}$$

with obvious symmetries $f(x,y) = f(-x,-y) = f(y,x)$.

One may check that $f_* \otimes f_*$ with $f_*(x) = \sqrt{\Lambda/2\pi}\exp(-\Lambda x^2/2)$ and $\Lambda = \lambda/2 + \sqrt{\lambda^2/4-1}$ does indeed solve (19). Unfortunately neither (18) nor (19) has a uniqueness statement. Instead, we make due with the following.

Lemma 4. Denote $\Lambda = \lambda/2 + \sqrt{\lambda^2/4-1}$. The integral equation (18) has a unique solution $f = f_*$ satisfying the side condition $\int_{-\infty}^{\infty}\exp(x^2/2\Lambda) \times f(x)\,dx < \infty$ given by $f_*(x) \equiv \sqrt{\Lambda/2\pi}\exp(-\Lambda x^2/2)$. However, it admits infinitely many solutions of less rapid decay. The analogous statement holds for (19) and the product $f_* \otimes f_*$.

Given that the "correct" solution has the best decay we can then verify the Proposition and thus the limit of the scaled one dimensional $\mathbf{M}_L$ marginal.

Proof of Lemma 3. The distribution function $\mu_{L^2}(\sigma_{L^2/2} \leqslant x)$ is given by the integral:

$$\frac{1}{z_{L^2}} \int_{\{\sum_{-L^2/2}^{L^2/2} \sigma_i^2 = 1,\, \sigma_{L^2/2} \leqslant x\}} \exp\left[DL^3 \left\{ H_{L^2-1}(\sigma) + \frac{D}{4} \sigma_{L^2/2}^4 + \sigma_{L^2/2}\sigma_{L^2/2-1} \right\} \right] d\sigma$$

$$= \frac{1}{z_{L^2}} \int_{-1}^{x} (1-z^2)^{\frac{L^2-3}{2}} \exp\left(L^3 \frac{D^2}{4} z^4 \right) \int_{\sum_{-L^2/2}^{L^2/2-1} \sigma^2 = 1} \exp[DL^3 H_{L^2-1}(\sigma)]$$

$$\times \exp\left[DL^3 \left(-\frac{1}{2} z^2 \lambda_{L^2-1}(\sigma) + z\sqrt{1-z^2}\, \sigma_{L^2/2} + \frac{D}{4} z^4 \sum \sigma_i^4 \right) \right] dz\, d\sigma$$

with the obvious notation H_{L^2-1} referring to the ensemble with one less particle. The density of interest then satisfies

$$f_{L^2}(x) = \mu_{L^2}(L^{3/2}\sqrt{D}\, \sigma_{L^2/2} = x)$$

$$= \frac{z_{L^2-1}}{\sqrt{D}\, L^{3/2} z_{L^2}} \mathbf{E}^{\mu_{L^2-1}}[e^{-\lambda_{L^2-1}(\sigma)\, x^2/2 + L^{3/2}\sqrt{D}\, x\sigma_{L^2/2-1}}](1 + \mathrm{O}(1/L))$$

for $L \uparrow \infty$. This, in turn, is schematized as

$$f_{L^2}(x) \simeq f_{L^2}(0) \exp(-\lambda x^2/2) \int_{-\infty}^{\infty} \exp(xz)\, f_{L^2-1}(z)\, dz, \tag{20}$$

where the fact that (Lemma 1) $\lambda_{L^2-1}(\sigma)$ converges in law to a constant $\lambda_\infty = \lambda$ is used.

By Lemma 2, $\limsup_{L \uparrow \infty} \mathbf{E}^{\mu_{L^2}}[\exp(L^{3/2}\gamma\sigma_{L^2/2})] < \infty$ for any γ, so, for fixed x, $\int e^{xy} f_{L^2}(y)\, dy$ converges along a subsequence. It then follows that $f_{L^2}(0) = L^{-3/2} z_{L^2-1} / (\sqrt{D}\, z_{L^2})$ is bounded both above and away from 0 for $L \uparrow \infty$. Differentiating (20) and re-runing the above argument provides a uniform bound on $|f'_{L^2}(x)|$. The conclusion is that there is a sequence over which the densities f_{L^2} themselves converge (uniformly on compacts) yielding the advertised integral equation. The derivation for the pair density is much the same. ▮

Proof of Lemma 4.[8] The proof is made in the one dimensional setting (18), it is the same for (19). Integrating (18) produces $1 = f(0)\sqrt{2\pi/\lambda} \times \int_{-\infty}^{\infty} \exp(x^2/2\lambda)\, f(x)\, dx$, providing some control of the tails of f. A sharper control is easily obtained.

[8] H. P. McKean helped with this.

Introduce the sequence $\Lambda_n = \lambda - \Lambda_{n-1}^{-1}$ where $\Lambda_1 = \lambda$ and $\Lambda_\infty = \lambda/2 + \sqrt{\lambda^2/4 - 1} \equiv \Lambda$. Next multiply both sides of (18) by $\exp(x^2/2\Lambda_n)$ and integrate to find

$$\int_{-\infty}^{\infty} \exp\left(\frac{x^2}{2\Lambda_n}\right) f(x)\, dx = f(0)\sqrt{2\pi/\Lambda_{n+1}} \int_{-\infty}^{\infty} \exp\left(\frac{x^2}{2\Lambda_{n+1}}\right) f(x)\, dx. \tag{21}$$

That is, $\exp(x^2/2\Lambda_n)\, f(x) \in L^1$ for any $n < \infty$, and we will show that solutions of (18) split into two classes depending upon whether the monotone limit

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \exp\left(\frac{x^2}{2\Lambda_n}\right) f(x)\, dx = \int_{-\infty}^{\infty} \exp\left(\frac{x^2}{2\Lambda}\right) f(x)\, dx \tag{22}$$

is finite or not. This is achieved by iteration. The nth iterate of (18) has the form

$$\begin{aligned} f(x) &= f(0)\exp\left(-\frac{\Lambda_n}{2}x^2\right) \frac{\int_{-\infty}^{\infty} \exp(xz/\Lambda_1\Lambda_2\cdots\Lambda_{n-1})\exp(z^2/2\Lambda_{n-1})\, f(z)\, dz}{\int_{-\infty}^{\infty} \exp(z^2/2\Lambda_{n-1})\, f(z)\, dz} \\ &\equiv f(0)\exp\left(-\frac{\Lambda_n}{2}x^2\right)\int_{-\infty}^{\infty} \exp(xz)\, d\mu_n(z). \end{aligned} \tag{23}$$

Now, if $\int \exp(\frac{x^2}{2\Lambda})\, f(x)\, dx < \infty$, taking limits on both sides of (21) implies $f(0) = \sqrt{\Lambda/2\pi}$. From here, (23) would provide the inequality $f(x) \geqslant \sqrt{\Lambda/2\pi}\, \exp(-\Lambda_n x^2/2)$ and so also $f(x) \geqslant f_*(x)$. Since both f and f_* are probability densities, the conclusion is that $f = f_*$.

If on the other hand $\int \exp(x^2/2\Lambda)\, dx = \infty$, we consider (23) in the limit:

$$f(x) = f(0)\exp(-\Lambda x^2/2)\int_{-\infty}^{\infty} \exp(xz)\, d\mu(z), \tag{24}$$

where $\mu = \mu_\infty$, the convergence of the latter being plain. From (24) we may infer the rules

$$f(x) = \frac{\exp(-\Lambda x^2/2)}{\sqrt{2\pi/\Lambda}} \circ d\mu(x) = f_*(x) \circ d\mu(x) \tag{25}$$

($\circ$ = convolution) and

$$d\mu(x) = \frac{\exp(-x^2/2\Lambda)}{f(0)\sqrt{2\pi/\Lambda}}\, d\mu(x/\Lambda). \tag{26}$$

To verify (25), apply our integral operator to $f_* \circ d\mu$. For (26), use (24) to express the transform $\int e^{xz}\, d\mu(z)$ and then note that (25) simplifies the remaining expression.

These rules allow us to generate infinitely many solutions of (18). By (25), $\mu[0] > 0$ is equivalent to $f(0) = \sqrt{\Lambda/2\pi}$ which implies $f = f_*$. Therefore, let μ have positive mass at some point $q \neq 0$. Since f is even $\mu[q] = \mu[-q]$, (26) implies both $\pm\Lambda q$ and $\pm q/\Lambda$ have positive μ mass. This proliferates: μ has a sequence of atoms at $\pm\Lambda^n q$ and $\pm q/\Lambda^n$ for $n \geqslant 1$ and one finds

$$d\mu(x) \equiv d\mu_q(x) = \mu[q] \sum_{n=1}^{\infty} \sum_{\pm} \omega_n \delta(x \pm \Lambda^n q) + \mu[q] \sum_{n=1}^{\infty} \sum_{\pm} \bar{\omega}_n \delta(x \pm \Lambda^{-n} q), \tag{27}$$

where

$$\omega_n = \frac{\exp[-\frac{q^2}{2} \frac{\Lambda^{2n+1}}{\Lambda - 1/\Lambda}]}{f(0)\sqrt{2\pi/\Lambda}} \qquad \text{and} \qquad \bar{\omega}_n = \frac{\exp[\frac{q^2}{2} \frac{\Lambda^{-2n+1}}{\Lambda - 1/\Lambda}]}{f^{-1}(0)\sqrt{\Lambda/2\pi}}.$$

It is readily checked that $\mu_q[\mathbb{R}] < \infty$, so $f_* \circ \mu_q$ solves (18). Even more solutions arise from convex combinations of μ_q's. The proof is finished. ▮

Proof of Proposition 1. The proof hinges on an integrability condition enforced by the ensemble $\mathbf{M}_L$. Recall the scaled marginal density can be written

$$\begin{aligned} &\mathbf{M}_L[\sqrt{L}\, Q(0) = a] \\ &\quad \simeq \frac{\mathbf{M}_L[0]}{\sqrt{L}}\, E^{\mu_{L^2}}[\exp\{-\lambda_{L^2} a^2/2 + L^{3/2}\sqrt{D}(\sigma_{-L^2/2} + \sigma_{L^2/2})\, a\}] \end{aligned}$$

up to multiplicative errors $1 + O(1/L)$, provided $|a| \leqslant K\sqrt{L}$ for a large K. Now for D large $L^{-1/2}\mathbf{M}_L[0]$ is bounded below, and integrating the last display in a over an appropriate range tending to the whole line as $L \uparrow \infty$ implies that any limiting density $f(x, y)$ must also satisfy

$$\int_{-\infty}^{\infty} \exp(-\lambda a^2/2) \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(ax + ay)\, f(x, y)\, dx\, dy\, da < \infty. \tag{28}$$

This holds when $f(x, y) = f_*(x)\, f_*(y)$, and next we show this is the only f for which it does.

As in Lemma 4, any potential limiting density satisfies $f(x, y) = [f_*(x)\, f_*(y)] \circ d\mu(x, y)$ with

$$d\mu(x, y) = \frac{\exp(-(x^2+y^2)/2\Lambda)}{f(0, 0)\, 2\pi/\Lambda}\, d\mu(x/\Lambda, y/\Lambda). \tag{29}$$

Also, $f(x, y) = f_*(x)\, f_*(y)$ is equivalent to $\mu = \delta_{(0,0)}$, this being the most rapid decay possible. A different solution, $f_{p,q} = f_* f_* \circ \mu_{p,q}$, may be formed by placing positive mass at a point (p, q) off the origin. Using the rule (29) to fill out the measure $\mu_{p,q}$ and the symmetries which any $f_{L^2}(x, y)$ must possess, one concludes that

$$\begin{aligned} f_{p,q}(x, y) = \frac{1}{z} \Bigg\{ & \sum_{n=1}^{\infty} \omega_n \left[e^{-\frac{1}{2}\Lambda |(x,y)+\Lambda^n(p,q)|^2} + e^{-\frac{1}{2}\Lambda |(x,y)-\Lambda^n(q,p)|^2} \right] \\ & + \sum_{n=1}^{\infty} \bar{\omega}_n \left[e^{-\frac{1}{2}\Lambda |(x,y)+\Lambda^{-n}(p,q)|^2} + e^{-\frac{1}{2}\Lambda |(x,y)-\Lambda^{-n}(q,p)|^2} \right] \Bigg\} \end{aligned} \tag{30}$$

where $\omega_n = r^n c_n$, $\bar{\omega}_n = r^{-n} c_{-n}^{-1}$ with $c_n = \exp(-\Lambda^{2n}(p^2+q^2)/2(\Lambda - 1/\Lambda))$, $r = 2\pi f(0, 0)/\Lambda < 1$, and z normalizes μ. Next let $pq > 0$ and compute: with $\Gamma_n(x, y) = (x - \Lambda^n p)^2 + (y - \Lambda^n q)^2$,

$$\begin{aligned} & \int_{-\infty}^{\infty} e^{-\lambda a^2/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(ax+ay)\, f_*(x)\, f_*(y) \circ \mu(dx, dy) \\ & \geqslant \frac{1}{z} \sum_{n=1}^{\infty} \omega_n \int_{-\infty}^{\infty} e^{-\lambda a^2/2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp(ax+ay) \exp\left[-\frac{\Lambda}{2} \Gamma_n(x, y) \right] \frac{dx\, dy}{(2\pi/\Lambda)}\, da \\ & = \frac{1}{z} \sum_{n=1}^{\infty} r^{-n} \exp(\Lambda^{2n} pq/(\Lambda - 1/\Lambda)) = \infty \end{aligned}$$

since $\Lambda > 1$. The argument with $pq \leqslant 0$ is similar. It only remains to note that it is sufficient to consider measures generated by a single point mass. But if (28) fails for such a measure, it fails for convex combinations and so, by approximation, also for f's with μ absolutely continuous. ∎

4. LIMITING JOINT DENSITY

We can now establish the limit of the scaled density (5). In terms of the p's this object splits into two components, recall (16). The asymptotics of the first piece may be obtained by straightforward stationary phase considerations. In fact, the particulars of the computation are reminiscent of an old work of Berlin and Kac.[1] For this reason, the result is stated without proof, after which the remaining details behind Theorem 1 are provided.

Lemma 5. For fixed a, b, I, and $L \uparrow \infty$,

$$\frac{1}{L^{3/2}}\, p\left(x, \frac{a}{\sqrt{L}}, \frac{b}{\sqrt{L}}, \frac{I}{L}+\frac{a^2}{L^2}\right) \simeq \frac{x^{3/2}}{8\pi I^2}\,[\phi]^{-1/2} \exp\left[-\frac{1}{2}(a^2+b^2)\left(1+\frac{x}{2I}-\phi\right)\right]$$
$$\times L^{1/2} \exp\left[-L\left\{I - I\phi + \frac{x}{2}\ln\left(\frac{x}{4I}+\frac{1}{2}\phi\right)\right\}\right]$$

with $\phi = \phi(x, I) = \sqrt{1+\frac{x^2}{4I^2}}$.

Proof of Theorem 1. Recall the second piece of the density in (16):

$$\frac{1}{\sqrt{L}\,\mathbf{Z}_L}\, p\left(L-x, \frac{b}{\sqrt{L}}, \frac{a}{\sqrt{L}}, DL - \frac{I}{L} - \frac{a^2}{L}\right)$$
$$= \frac{p(L, 0, 0, LD)}{\sqrt{L}\, Z_L} \times \frac{(2\pi)^{Lx/2}\, p(L-x, 0, 0, LD)}{(DL^3)^{Lx/2}\, p(L, 0, 0, LD)}$$
$$\times \mathbf{E}^{\mu_{L^2-Lx}}\left[\exp\left\{\frac{1}{2}(LI + a^2 + b^2)\,\lambda_{L^2-Lx}\right.\right.$$
$$\left.\left. + L^{3/2}\sqrt{D}\,(a\sigma_{-(L^2-Lx)/2} + b\sigma_{(L^2-Lx)/2})\right\}\right]$$
$$\equiv \Gamma_L \times \mathbf{E}^{\mu_{L^2-Lx}}[\text{etc.}]$$

up to asymptotically small errors. Next combine this with the result of Lemma 5:

$$\mathbf{M}_L\left[\sqrt{L}\,Q_0 = a, \sqrt{L}\,Q_{Lx} = b, \sum_1^{Lx-1}[\sqrt{L}\,Q_k]^2\,\frac{1}{L} = I\right]$$
$$\simeq \Gamma_L\, \frac{x^{3/2}}{4\pi I^2}\left(1+\frac{x^2}{4I^2}\right)^{-1/4} \exp\left[-\frac{1}{2}(a^2+b^2)\left(\Lambda + \frac{x}{2I} - \sqrt{1+\frac{x^2}{4I^2}}\right)\right]$$
$$\times \sqrt{L}\exp\left[-L\left\{\frac{\lambda}{2} I - I\sqrt{1+\frac{x^2}{4I^2}} + \frac{x}{2}\ln\left(\frac{x}{2I}+\sqrt{1+\frac{x^2}{4I^2}}\right)\right\}\right]$$
$$\times \mathbf{E}^{\mu_{L^2-Lx}}\left[\exp\left\{\frac{1}{2}LI(\lambda - \lambda_{L^2-Lx,\,D}(\sigma)) + \frac{1}{2}(a^2+b^2)(\Lambda - \lambda_{L^2-Lx}(\sigma))\right\}\right.$$
$$\left. \times \exp\{L^{3/2}\sqrt{D}\,(a\sigma_{-(L^2-Lx)/2} + b\sigma_{(L^2-Lx)/2})\}\right]$$
$$\equiv \Gamma_L \Phi(I, x)\left[\frac{\rho_{x,I}}{\pi}\exp[-\rho_{x,I}(a^2+b^2)]\right][\sqrt{L}\exp[-L\Psi(I)]]\; E[a, b, I]. \tag{31}$$

for fixed a, b, and I and $L \uparrow \infty$. Here $\lambda = \lambda_{\infty, D}$ and $\Lambda = \lambda/2 + \sqrt{\lambda^2/4 - 1}$. In (31) we have normalized in the a, b variables in that $G_I(a)\, G_I(b) \equiv (\rho_{x,I}/\pi) \exp[-\rho_{x,I}(a^2+b^2)]$ is the density of two independent centered Gaussians with mean-square one over $\Lambda + \frac{x}{2I} - \sqrt{1 + \frac{x^2}{4I^2}}$.

Regarding the I variable, the function

$$\Psi(I) = \frac{\lambda}{2} I - I \sqrt{1 + \frac{x^2}{4I^2}} + \frac{x}{2} \ln\left(\frac{x}{2I} + \sqrt{1 + \frac{x^2}{4I^2}}\right)$$

is strictly convex with minimum at $I_0 \equiv (x/2)(\lambda^2/4 - 1)^{-1/2}$; it follows that the measure

$$\mathscr{I}_L(I)\, dI \equiv \sqrt{L\Psi''(I_0)/2\pi}\, \exp[-L[\Psi(I) - \Psi(I_0)]]\, dI$$

converges weakly to the unit mass at I_0. At $I = I_0$, $\Psi(I_0) = (x/2) \ln \Lambda$, and $\rho_{x, I_0} = \sqrt{\lambda^2/4 - 1} = \rho_D$, the limiting mean square in the statement.

Summarizing, the measure of interest has the form:

$$\begin{aligned} \mathbf{M}_L[da, db, dI] = \Lambda^{-\frac{Lx}{2}} \Gamma_L[\Phi(I) \sqrt{2\pi/\Psi''(I)}]\, E[a, b, I] \\ \times G_I(a)\, G_I(b)\, \mathscr{I}_L(I)\, da\, db\, dI, \end{aligned} \tag{32}$$

where $G_I(a)\, G_I(b)\, \mathscr{I}_L(I)\, da\, db\, dI$ converges to the advertised limiting measure. It remains to explain why the other factors fall into line for $I = I_0$ and $L \uparrow \infty$. That $E[a, b, I] \simeq 1$ for $L \uparrow \infty$ follows from $\lim_{L \uparrow \infty} \mathbf{E}^{\mu_{L^2}}[\exp L^{3/2} \sqrt{D}(a\sigma_{-L^2/2} + b\sigma_{L^2/2})] = \exp[(a^2+b^2)/2\Lambda]$ (Proposition 1) along with the relation $\Lambda + \Lambda^{-1} = \lambda$. Next, one computes that $\sqrt{\Psi''(I_0)}/\Phi(I_0) = \sqrt{\rho_D}$ yielding the constant factor

$$\begin{aligned} \frac{\sqrt{\pi}}{\rho_D} \Lambda^{-\frac{Lx}{2}} \Gamma_L &= \frac{\mathbf{M}_L[0]}{\sqrt{L}} \frac{\sqrt{\pi}}{\rho_D} \times \frac{(2\pi)^{Lx/2}\, p_L(L-x, 0, 0, LD)}{(DL^3)^{Lx/2}\, \Lambda^{Lx/2} p_L(L, 0, 0, LD)} \\ &= \frac{\mathbf{M}_L[0]}{\sqrt{L}} \frac{\sqrt{\pi}}{\rho_D} \times \prod_{k=0}^{Lx-1} \frac{\sqrt{2\pi}\, z_{L^2-k-1}}{\sqrt{D\Lambda}\, L^{3/2} z_{L^2-k}} \end{aligned} \tag{33}$$

which is to equal one in the limit. Now, from Proposition 1 we have that $\mathbf{M}_L[0]/\sqrt{L} \to \sqrt{\rho_D/\pi}$ and a look at the proof will explain that also $\sqrt{D\Lambda/2\pi}\, L^{3/2} z_{L^2} z_{L^2-l}^{-1} \geqslant 1$ for $L \gg 1$. In other words, the entirety of (33) is asymptotically less than one, and so any limiting $\mathbf{M}_L[da, db, dI]$ is dominated by the claimed limiting distribution. However, having already shown that $\mathbf{M}_L[\sqrt{L}\,Q_0 \in da]$ converges, the rotation invariance of the scaled ensemble implies tightness of the present joint density and an inequality suffices. The proof is finished. ∎

APPENDIX A

Proof of Lemma 1. The stated level of concentration of the $\mu_{L^2,D}$ about the H-maximimizers follows from simple estimates; the details are not reported.

(a) $H_{L^2,D}$ is evaluated at the appropriate test function $\{\sigma\}$. For $D \ll 1$, take $\sigma_k = \theta^{-1} q^{|k|}$ where $q < 1$ and $\theta = (1 + 2\sum_1^n q^{2k})^{1/2}$. For large D a convenient choice is $\sigma_k = D^{-|k|}$ for $|k| \geqslant 1$ with $\sigma_0^2 = 1 - 2\sum_1^n D^{-2k}$.

(b) First, it is clear that at maximum all the σ_k are of one sign, so from here on we take them positive. The task is to maximize

$$H_{L^2}(\sigma) = \frac{D}{4}\sum_i \sigma_i^4 + \frac{1}{2}\sum_{i,j} \Gamma_{i,j}\sigma_i\sigma_j$$

on $\sum \sigma_k^2 = 1$ for $\Gamma_{i,j} = 1$ if $|i-j| \leqslant 1$ and 0 otherwise. Both the sum of fourth powers and the constraint are insensitive to permutations of the indices. The term $\sum \Gamma_{i,j}\sigma_i\sigma_j$ responds to a rearrangement theorem of Riesz:[9] it is largest when $\{\sigma\}$ peaks at the center ($n = 0$) and is increasing/decreasing to the left/right.

This increasing/decreasing immediately gives $\sigma_{|k|} \leqslant |k+1|^{-1/2}$. This, along with the Euler–Lagrange equations ($\sigma_{k-1} + \sigma_{k+1} = \lambda\sigma_k - D\sigma_k^3$) gives $\sigma_{L^2-1} \geqslant (\lambda - D/(L^2+1))\,\sigma_{L^2}$. Upon iteration this is

$$\sigma_{L^2-k} \geqslant \left(\lambda - \frac{D}{L^2+1}\right) \prod_{j=1}^{k} \left(\lambda - 1 - \frac{D}{L^2+1-j}\right) \sigma_{L^2}.$$

Now, for fixed D and large enough L we have from (a) that $\lambda > 2 + \delta$ for $\delta > 0$ and it is easy to conclude that there are constants M, C, and θ such that $|k| > M$ implies $\sigma_{|k|} \leqslant Ce^{-\theta|k|}$. Here θ increases with D.

An extremal σ^∞ is furnished by the limit of the maximizers σ^L of H_{L^2}. The proven decay of the tails of σ^L yields both s limiting subsequence and the necessary domination: $H_\infty(\sigma^\infty) = m_\infty$.

(c) As for uniqueness, large D can be handled due to sharper estimates for H and λ. In detail, the lower bound for large D along with the

[9] See ref. 4. We thank Professor Elliot Lieb for pointing this out to us.

fact that $\sigma_0 \geqslant \sigma_k$ at maximum yields that there $(D/4)+(3/2)\,D \leqslant H(\sigma) \leqslant (D/4)\,\sigma_0^2+1$. This, in turn, implies $\sum_{k\neq 0}\sigma_k^2 = 1-\sigma_0^2 \leqslant 4/D$, so that $\sigma_{\pm 1} \leqslant 4/D$ and in fact $\sigma_{\pm|k|} \leqslant \text{const.} \times D^{-|k|}$. In brief, any maximizer satisfies $\sigma_0 = 1-\mathrm{O}(1/D)$.

Suppose there are two maximizers (σ^0 and σ^2). Think of H as a function of the angle $\theta\colon 0 \leqslant \theta \leqslant \theta_1$ along the geodesic between these points: $\sigma^0 = \sigma(0)$ and $\sigma^1 = \sigma(\theta_1)$ with $h(\theta) = H(\sigma(\theta))$. Look at the action of a small rotation along this arc upon h and compute: $\sigma''(\theta) = -\sigma(\theta)$, $\|\sigma'(\theta)\| = 1$, and

$$h''(\theta) = -\lambda + 3D \sum \sigma_k^2 (\sigma_k')^2 + 2 \sum \sigma_k' \sigma_{k+1}' \leqslant -\lambda(\theta) + 3D(\sigma_0')^2\,(\theta) + \mathrm{O}(1).$$

The inequality follows from Cauchy–Schwartz along with the bound $\sigma_{\pm|k|} \leqslant 4/D$ for $|k| \geqslant 1$. Along the arc $\sum_{k\neq 0}\sigma_k^2 \simeq 4/D$ maintains, and this implies that $\lambda(\theta) \geqslant D\sigma_0^4(\theta) \simeq D(1-\mathrm{O}(1/D))$ and $(\sigma_0')^2\,(\theta) \simeq \sin^2(\theta) \simeq \mathrm{O}(1/D)$. This forces the needed contradiction that $h''(\theta) < 0$ once D is large enough. The proof is finished. ∎

Proof of Lemma 2. Wanting to control $\mathbf{E}^{\mu_L^2}[\exp L^3 \gamma \sigma_{L^2}^2]$ we may assume from (13) that $\sigma_{L^2} < L^{-5/6}$ say. Next, repeating the calculation starting the proof of Lemma 3: with some constant C and $L \uparrow \infty$,

$$\mathbf{E}^{\mu_L^2}[e^{L^3\gamma\sigma_{L^2}^2}] \leqslant \frac{C}{z_{L^2}} \int_{S_1^{L^2-2}} d\sigma \exp[L^3 D H_{L^2-1}(\sigma)]$$
$$\times \int_{-\infty}^{\infty} d\sigma_{L^2} \exp\left[L^3\gamma\sigma_{L^2}^2 - L^3 D \frac{1}{2} \lambda_{L^2-1}(\sigma)\, \sigma_{L^2}^2 + L^3 D \sigma_{L^2}^2 \right]$$

where we have also restricted to the set $\sigma_{L^2-1} \leqslant \sigma_{L^2}$. The Gaussian integration in σ_{L^2} may then be performed if γ is chosen so that $D\lambda/2 - D - \gamma > 0$ for $L \uparrow 0$ which may be done the results of Lemma 1(a). The right side is then bounded via the simple estimate seen before: $z_{L^2} \geqslant \text{const.}\, L^{-3/2} z_{L^2}$. For the integral over the set $\sigma_{L^2} < \sigma_{L^2-1}$ one simply iterates the same argument to complete the proof.

As for $L^{-1/2}\mathbf{M}_L[0]$, we first mention that one has an upper bound for all D. Examining the inverse, a little manipulation will show $\sqrt{L}\,\mathbf{M}_L[0]^{-1} =$

$$\frac{\sqrt{L}\,\mathbf{Z}_L}{p(L,0,0,LD)} \geqslant D^{1/2} L^{3/2} \int_{-\delta}^{\delta} (1-c^2)^{(L^2-3)/2}\, \mathbf{E}^{\mu_L^2}[\exp[-L^3 D \lambda_{L^2}(\sigma)\, c^2/2]]$$
$$\times \exp[DL^3 (c(1-c^2)^{1/2}\, (\sigma_{-L^2/2} + \sigma_{L^2/2}))]]\, dc,$$

for any $\delta > 0$ but less than 1. Next, Jensen's inequality is applied in the μ_{L^2} expectation to produce: with $\mathbf{E}^{\mu_{L^2}}[\sigma_k] = 0$ for any k,

$$\sqrt{L}(\mathbf{M}_L[0])^{-1} \geqslant \sqrt{\frac{2\pi}{\mathbf{E}^{\mu_{L^2}}[\lambda_{L^2}]}}\,(1-\delta^2)^{L^2/2}\left\{1 - \frac{1}{\delta}\,\frac{\exp(-DL^3\mathbf{E}^{\mu_{L^2}}[\lambda_{L^2}]\,\delta^2)}{L^{3/2}D^{1/2}\mathbf{E}^{\mu_L^2}[\lambda_{L^2}]}\right\}.$$

Here we have used the fact $\mathbf{E}^{\mu_{L^2}}[\lambda_{L^2}(\sigma)] \geqslant 2\mathbf{E}^{\mu_{L^2}}[H_{L^2}(\sigma)] > 0$ for L large. Taking $\delta = \delta(L) = L^{-3/4}$ gives $\limsup_{L\uparrow\infty} \mathbf{M}_L[0]/\sqrt{L} \leqslant (1+D/2)/\sqrt{2\pi}$.

An estimate of $L^{-1/2}\mathbf{M}_L[0]$ from below requires more. A necessary condition stems from integrating the density $\mathbf{M}_L[Q_0 = a]$ (recall (14)) over some large range $|a| \leqslant K$. By tightness we find that

$$1 - \mathrm{o}(1) \leqslant \frac{\mathbf{M}_L[0]}{\sqrt{L}}\,\mathbf{E}^{\mu_{L^2}}\left[\sqrt{\frac{2\pi}{\lambda_{L^2}}}\,\exp[L^3D(\sigma_{-L^2/2}+\sigma_{L^2/2})^2/2\lambda_{L^2}]\right],$$

up to small errors on the right hand side for L large. Thus, $L^{-1/2}\mathbf{M}_L[0] = O(1)$ for $L \uparrow \infty$ is implied by

$$\limsup_{L\uparrow\infty}\,\mathbf{E}^{\mu_{L^2}}[e^{L^3D(\sigma_{-L^2/2}+\sigma_{L^2/2})^2/2\lambda_{L^2}}] \leqslant \limsup_{L\uparrow\infty}\,\mathbf{E}^{\mu_{L^2}}[e^{2L^3D\sigma^2_{L^2/2}/\lambda_{L^2}}] < \infty;$$

allowing the desired conclusion for large D by taking $\gamma = 2$ above since in that case we know $\lambda_{L^2,D} > D$ for $L \uparrow \infty$. ∎

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